Universidade de Lisboa
Instituto Superior de Economia e Gestão
Bsc in Economics, Finance and Management

Mathematics I – Semester 2 - 2025/2026

Regular Assessment – 13th of May 2026

Duration: $(120 + \varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name ....

ID Student #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(8)* Concerning the set

$$S = \left\{1 - \frac{1}{2^n} : n \in \mathbb{N}\right\} \cup \,]2, 3],$$

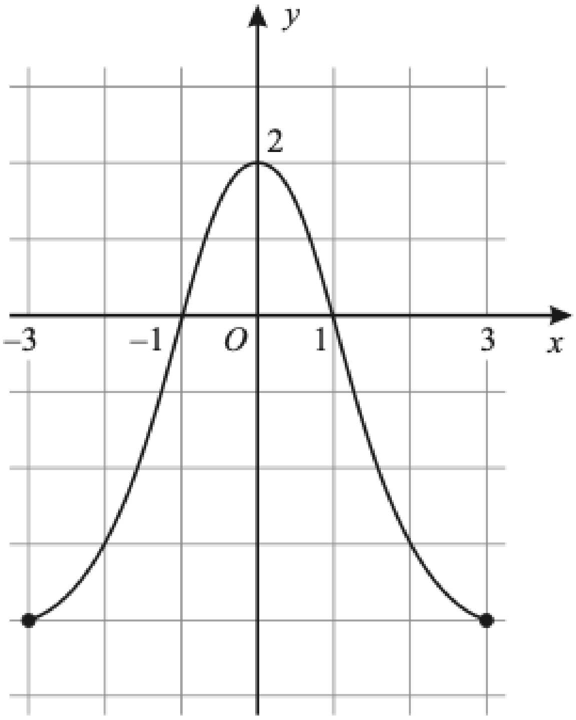
we may conclude that:

- $\operatorname{int}(S) =$ .... (interior of S)
- The set of isolated points of S is ....
- $\inf S =$ .... (infimum of S; note that $0 \notin \mathbb{N}$)
- The set S is **not** compact set because it is not ....

(b) *(10)* Let $(v_n)_{n\in\mathbb{N}}$ be a **geometric progression** of negative terms such that $v_2 = -8$ and $v_4 = -2$. Then:

- The ratio of $(v_n)_{n\in\mathbb{N}}$ is $r =$ .... and $v_1 =$ ....
- The sequence $(v_n)_{n\in\mathbb{N}}$ is monotonic .... and $\lim v_n =$ ....
- $\sum_{n=3}^{+\infty} v_n =$ ....

(c) *(8)* Let $f : [-3,3] \to \mathbb{R}$ the map whose graphical representation is below. Integer objects have integer images.



Let $g : [e^{-3}, e^3] \to \mathbb{R}$ be the map whose analytical expression is $g(x) = \ln x$. Then:

- Im(f) = ... (Image of f)
- the map $f \circ g$ has **two** zeros: .... and ....
- $\lim_{n\to+\infty} f\left(-\frac{1}{n}\right) =$ ....

(d) *(6)* Consider the differentiable map $f : \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} f(x) + x^2[f(x)]^3 = 10, & \forall x \in \mathbb{R} \\ f(1) = 2. \end{cases}$$

Then

$$f'(1) = \text{..............}$$

(e) *(6)* Let $f : [1,4] \to \mathbb{R}$ be a differentiable function such that $f(1) = 5$ and

$$\forall x \in]1,4[, \quad 2 \le f'(x) \le 6.$$

From the **Lagrange Mean Theorem**, we may conclude that:

$$f(4) \ge \text{..........}$$

(f) *(12)* Regarding the 2π–periodic map $f(x) = e^{\cos(x)-1}$, $x \in \mathbb{R}$, one may conclude that:

- the quadratic Taylor polynomial $P_2(x)$ centered at $x = 0$ is given by:

  $$P_2(x) = \text{..}$$

- using $P_2(x)$, the approximated value of $f(0.2)$ is:

  $$f(0.2) \approx \text{.................}$$

- if the maximum value of $|f'''(x)|$ on $[0, 0.2]$ is $M \approx 0.197$, then the upper bound for the absolute error is:

  $$|f(0.2) - P_2(0.2)| \le \text{..............................}$$

  (*apply the formula to the case under consideration;* ***not necessary*** *to compute*)

(g) *(8)* Consider the map $F : \mathbb{R} \to \mathbb{R}$ defined by:

$$F(x) = x^4 + \int_{x^2}^{5} e^{-t^2}\, dt$$

Hence:

- $F(\sqrt{5}) =$ ....
- $F'(x) =$ .... and $F''(0) =$ ....
- At $x = 0$, the function F has a relative .... (*choose between maximum/minimum*)

(h) *(12)* In the vector space $\mathbb{R}^3$, consider the vectors $\vec{u} = (1, -2, 2)$, $\vec{v} = (k, 1, 1)$, and $\vec{w} = (0, 3, -3)$, where $k \in \mathbb{R}$. Then:

- $\vec{u} + (\ldots, \ldots, \ldots) = 2\vec{w}$.
- If $k =$ .... then $\vec{u} \cdot \vec{v} = 0$.
- $\|\vec{w}\| =$ ....
- For all $k \in \mathbb{R}$, the vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are linearly ....

(i) *(9)* Consider the matrix $\mathbf{A} \in M_{3\times 3}(\mathbb{R})$ such that (*the letters represent real numbers*):

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad \det(\mathbf{A}) = -5.$$

We may conclude that:

- $\det(\mathbf{A}^T) =$ .... and $\det(3\mathbf{A}) =$ ....
- $\det \begin{bmatrix} a & b & c \\ 2d - 3a & 2e - 3b & 2f - 3c \\ g & h & i \end{bmatrix} =$ ....

(j) *(6)* In $\mathbb{R}^3$, consider the linear system $\mathbf{A}X = \mathbf{B}$, where $\mathbf{A} \in M_{3\times 3}(\mathbb{R})$ and $\mathbf{B} \in M_{3\times 1}(\mathbb{R})$. One knows:

- $\det(\mathbf{A}) = 0$ but there is at least one 2×2 submatrix with a non-zero determinant;
- $\mathbf{B} \neq \vec{0}$ is equal to the second column matrix of A.

Then:

- $r(\mathbf{A}|\mathbf{B}) =$ ....
- The system $\mathbf{A}X = \mathbf{B}$ may be classified as .... with degree of freedom equal to ....

Part II

- Give your answers in exact form.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. Using the **Principle of Mathematical Induction**, prove that for all $n \in \mathbb{N}$, we have:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

2. For $\beta \in \mathbb{R}$, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ \cos(x) + \beta x^2 & \text{if } x < 0 \end{cases}$$

(a) Show that f is **continuous** for any $\beta \in \mathbb{R}$.

(b) Determine β, if it exists, such that f is **differentiable** at $x = 0$.

(c) The restriction $f|_{[-5,5]}$ has an **absolute maximum**. Justify without performing computations.

3. Consider the rational function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by:

$$g(x) = \frac{x^2 + x + 4}{x^3 + 4x}.$$

(a) Find the equations of all the **asymptotes** of the graph of g.

(b) Compute

$$\int_2^{2\sqrt{3}} g(x)\, dx.$$

4. Consider the matrices $\mathbf{A}, \mathbf{B}^{-1} \in M_{3\times3}(\mathbb{R})$ defined by:

$$\mathbf{A} = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

where $k \in \mathbb{R}$. Note that $\det \mathbf{B}^{-1} \neq 0$.

a) Determine the values of k for which $\mathbf{A}$ is **singular**. What can you conclude about the linear independence of its row vectors for these values?

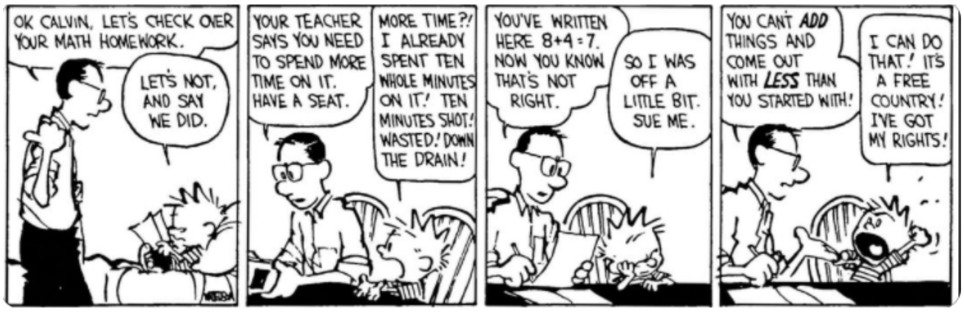
b) For $k = 0$, solve the matrix equation $\mathbf{BX} = \mathbf{A}$ for $\mathbf{X} \in M_{3\times3}(\mathbb{R})$.

5. Consider the following system of linear equations in the variables $x, y, z \in \mathbb{R}$, with parameters $k, m \in \mathbb{R}$:

$$\begin{cases} x + ky = 2 \\ kx + y = m \\ ky + z = 0 \end{cases}$$

a) Find the values of k for which the system has a **unique** solution.

b) For $k = 2$ and $m = 1$, solve the system using **Cramer's Rule**.



Scores:

I	II.1	II.2(a)	II.2(b)	II.2(c)	II.3(a)	II.3(b)	II.4(a)	II.4(b)	II.5(a)	II.5(b)
85	10	10	10	10	10	20	10	10	15	10